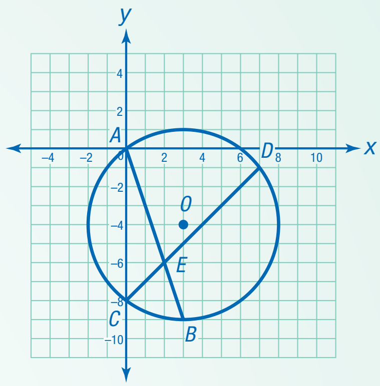
LESSON 29 Using Coordinates to Prove Theorems about Circles

Representing figures as algebraic equations can help you prove geometric theorems.

EXAMPLE A Demonstrate the intersecting chords theorem by proving that $(AE)(BE) = (CE)(DE)$.



1 Write an equation for each chord.

The endpoints of $\overline{AB}$ appear to be $A(0, 0)$ and $B(3, -9)$. Find the slope of $\overline{AB}$.

$$m_{\overline{AB}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-9 - 0}{3 - 0} = \frac{-9}{3} = -3$$

The y-intercept for $\overleftrightarrow{AB}$ is (0, 0). So, the equation of the line is $y = -3x + 0$.

The endpoints of $\overline{CD}$ appear to be $C(0, -8)$ and $D(7, -1)$. Find the slope of $\overline{CD}$.

$$m_{\overline{CD}} = \frac{-1 - (-8)}{7 - 0} = \frac{7}{7} = 1$$

The y-intercept for $\overleftrightarrow{CD}$ is $(0, -8)$. So, the equation of the line is $y = x - 8$.

2 Find the point of intersection.

The point of intersection is the solution to the system containing $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

$$\begin{cases} y = -3x \\ y = x - 8 \end{cases}$$

Substitute $-3x$ for y in $y = x - 8$.

$-3x = x - 8$

$-4x = -8$

$x = 2$

Substitute 2 for x in either equation.

$y = -3x = -3(2) = -6$

Point E is $(2, -6)$.

3 Show that $(AE)(BE) = (CE)(DE)$.

$AE = \sqrt{(2 - 0)^2 + (-6 - 0)^2} = \sqrt{40}$

$BE = \sqrt{(2 - 3)^2 + (-6 - (-9))^2} = \sqrt{10}$

$CE = \sqrt{(2 - 0)^2 + (-6 - (-8))^2} = \sqrt{8}$

$DE = \sqrt{(2 - 7)^2 + (-6 - (-1))^2} = \sqrt{50}$

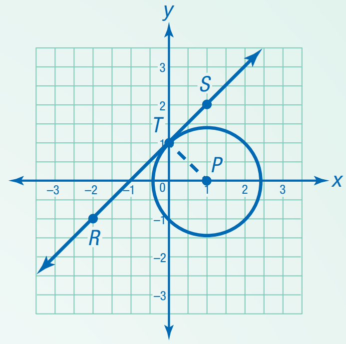
$(\sqrt{40})(\sqrt{10}) \stackrel{?}{=} (\sqrt{8})(\sqrt{50})$

$\sqrt{400} = \sqrt{400}$ ✓

TRY

Show that $A(0, 0)$, $B(3, -9)$, $C(0, -8)$, and $D(7, -1)$ lie on the circle whose center is $(3, -4)$ and whose radius is 5 units long.

EXAMPLE B The equation of circle P is $(x - 1)^2 + y^2 = 2$. The equation of line RS is $y = x + 1$. Prove that $\overleftrightarrow{RS}$ is tangent to circle P. Then prove that the tangent line is perpendicular to the radius of the circle.



1 Prove that $\overleftrightarrow{RS}$ is tangent to circle P.

A tangent line intersects a circle at exactly one point. So, if $\overleftrightarrow{RS}$ is tangent to circle P, then there will be only one solution to the system containing the equations of the line and the circle.

$$\begin{cases} (x - 1)^2 + y^2 = 2 \\ y = x + 1 \end{cases}$$

Substitute $x + 1$ for y in the equation of the circle and solve for x.

$$(x - 1)^2 + y^2 = 2$$
$$(x - 1)^2 + (x + 1)^2 = 2$$
$$x^2 - 2x + 1 + x^2 + 2x + 1 - 2 = 0$$
$$2x^2 = 0$$
$$x^2 = 0$$
$$x = 0$$

Substitute 0 for x in the linear equation.

$y = x + 1 = 0 + 1 = 1$

▶ There is only one real solution, (0, 1), so $\overleftrightarrow{RS}$ intersects circle P at this one point only. So, $\overleftrightarrow{RS}$ must be tangent to circle P.

2 Prove that radius $\overline{PT}$ is perpendicular to $\overleftrightarrow{RS}$.

Perpendicular lines have slopes that are opposite reciprocals of each other.

The equation for $\overleftrightarrow{RS}$ is $y = x + 1$, so, $m_{\overline{RS}} = 1$.

Find the slope of radius PT.

Point T was found to be (0, 1). Point P is the center of circle P. The equation of circle P is $(x - 1)^2 + y^2 = 2$, so the center, (h, k), of the circle is (1, 0).

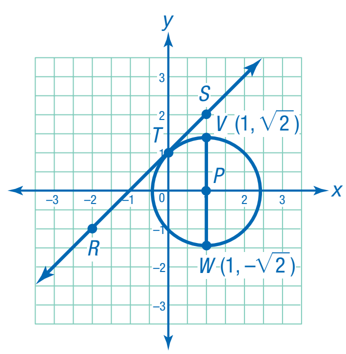
$$m_{\overline{PT}} = \frac{1 - 0}{0 - 1} = \frac{1}{-1} = -1$$

$$m_{\overline{RS}} \cdot m_{\overline{PT}} = 1 \cdot -1 = -1$$

▶ The slopes of $\overleftrightarrow{RS}$ and $\overleftrightarrow{PT}$ are opposite reciprocals, so $\overleftrightarrow{PT} \perp \overleftrightarrow{RS}$.

TRY

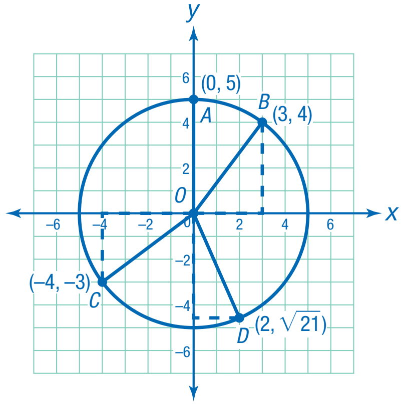
Demonstrate the Secant-Tangent Theorem by proving that $(SV)(SW) = (ST)^2$.



Practice

Use circle *O* to prove that all radii of a circle have the same length.

1. Find the length of radius *OA*.



$OA =$ ______________________ = __________

> **HINT** Since $\overline{OA}$ is a vertical segment, $OA = |y_2 - y_1|$.

2. Find the lengths of other radii.

$OB =$ ______________ = __________

$OC =$ ______________ = __________

$OD =$ ______________ = __________

3. Will all radii of the circle have the same length? Explain how you know.

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Prove or disprove.

4. The line $y = x - 1$ is tangent to the circle $x^2 + (y - 3)^2 = 8$ at the point (2, 1). Prove the tangent line is perpendicular to the radius at the point of tangency.

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> **REMEMBER** Perpendicular lines have slopes that are opposite reciprocals.

5. Prove or disprove that the point $(1, \sqrt{3})$ lies on the circle centered at the origin and containing the point (3, 0).

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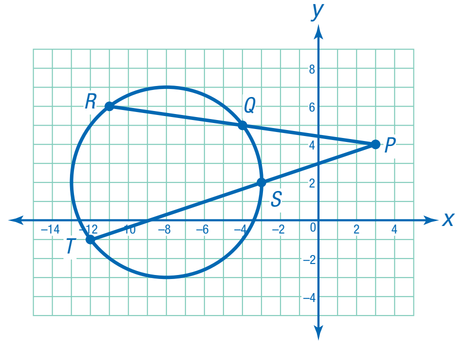
6. Line segments *PR* and *PT* are secants drawn from point *P*. Prove that the product of one secant segment and its external segment is equal to the product of the other secant segment and its external segment.



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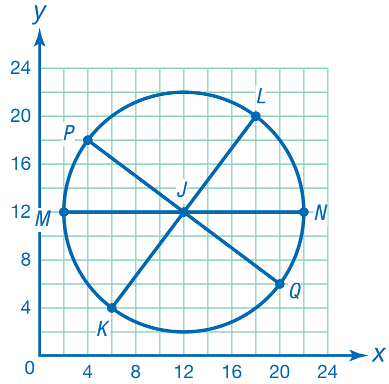
Prove or disprove.

7. **PROVE** Prove that the diameter of circle *J* is twice the length of its radius.



Use the distance formula (or another formula) to find each length. Show your work.

MN = ____________________________ = __________

MJ = ____________________________ = __________

KL = ____________________________ = __________

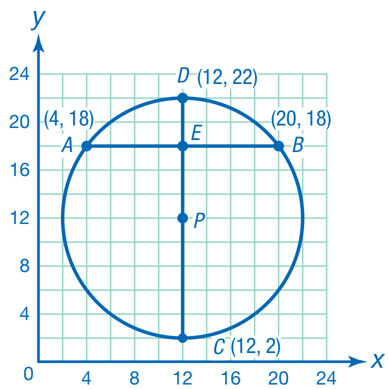
KJ = ____________________________ = __________

PQ = ____________________________ = __________

PJ = ____________________________ = __________

So, each diameter is __________ the length of each radius.

8. **JUSTIFY** In circle *P*, chord $\overline{CD}$ intersects chord $\overline{AB}$ at point *E*.



Show that because the diameter intersects the chord and is perpendicular to the chord, it bisects the chord. Justify each step by completing the two-column proof.

Statements	Reasons
1. $\overline{AB}$ and $\overline{CD}$ intersect at point *E*.	Given
2. $\overline{CD}$ is a diameter of circle *P*.	$\overline{CD}$ is a chord that passes through __________.
3. $\overline{AB} \perp \overline{CD}$	$\overline{AB}$ is a horizontal segment and ______________________
4. *AE* = *EB*	$AE = \lvert 12 -$ __________ $\rvert =$ __________ *EB* = ______________________
5. $\overline{CD}$ bisects $\overline{AB}$.	$\overline{CD}$ divides $\overline{AB}$ into two segments of equal lengths. Definition of ______________.